Weakly Arf rings

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based on the works jointly with

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AMS Special Session on Homological Methods in Algebra

October 4, 2020

1. Introduction

- In 1949, Cahit Arf explored the multiplicity sequences of curve singularities.
- In 1971, J. Lipman defined "Arf rings" for one-dimensional CM semi-local rings.

Definition 1.1 (Lipman)

Let R be a CM semi-local ring with $\dim R = 1$. Then R is called *an Arf ring*, if the following hold:

(1) Every integrally closed *open* ideal I has a principal reduction.

(2) If $x, y, z \in R$ s.t.

$$x \text{ is a NZD on } R \text{ and } \frac{y}{x}, \ \frac{z}{x} \in \overline{R},$$

then $yz/x \in R$.

Notice that

(1) $I^{n+1} = aI^n$ for $\exists\ n \geq 0$ and $\exists\ a \in I$.

(2) Stability of I (if reduction exists).

Hence

Theorem 1.2 (Lipman)

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is Arf} \iff \text{Every integrally closed open ideal is stable.}$$

When R is a CM local ring with $\dim R = 1$,

if R is an Arf ring, then R has minimal multiplicity.

We assume

- $(R, \mathfrak{m})$ is a Noetherian complete local domain with $\dim R = 1$
- $R/\mathfrak{m}$ is an algebraically closed field of characteristic 0

Lipman proved:

$$R \text{ is } \textit{saturated} \quad \Longrightarrow \quad R \text{ has minimal multiplicity.}$$

Moreover, among all Arf rings between R and $\overline{R}$,

$$\exists \text{ the smallest one } \mathrm{Arf}(R), \text{ called Arf closure.}$$

Lipman extends the results of C. Arf about multiplicity sequences.

Question 1.3

What happens if we remove the condition (1)?

Definition 1.4

A commutative ring R is said to be *weakly Arf*, provided

$$yz/x \in R, \text{ whenever } x, y, z \in R \text{ s.t. } x \in R \text{ is a NZD}, y/x, z/x \in \overline{R}.$$

Example 1.5

- $R = \overline{R}$
- $\mathrm{e}(R) \leq 2$, where R is a Noetherian local ring.
- $k[H]$, where H is an Arf semigroup
- $k[t^n, t^{n+1}, \dots, t^{2n-1}]$ $(n \geq 2)$

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2. Weakly Arf rings

Throughout this talk

- R a Noetherian ring
- $W(R)$ the set of NZDs on R
- $\Lambda(R) = \{\overline{(x)} \mid x \in W(R)\}$

Theorem 2.1

$$R \text{ is a weakly Arf ring} \iff \text{Every } I \in \Lambda(R) \text{ is stable.}$$

Recall $\Lambda(R) = \{\overline{(x)} \mid x \in W(R)\}$.

Example 2.2

Let k be a field and set $R = k[[X, Y]]/(XY(X+Y))$. Then

- R is a CM local reduced ring with $\dim R = 1$.
- $\mathfrak{m}$ does not have a principal reduction, if $k = \mathbb{Z}/(2)$.
- $\{\text{integrally closed } \mathfrak{m}\text{-primary ideals}\} = \{\mathfrak{m}\} \cup \{\text{stable ideals}\}$.

Hence, if $k = \mathbb{Z}/(2)$, then R is a weakly Arf ring, but not an Arf ring.

Proposition 2.3

Let $\varphi : R \to S$ be a homomorphism of rings. Suppose $aS \cap R = aR$ and $\varphi(a) \in W(S)$ for $\forall a \in W(R)$. If S is weakly Arf, then so is R.

Corollary 2.4

(1) *Let S be an integral domain, $R \subseteq S$ a subring of S s.t. R is a direct summand of S. If S is a weakly Arf ring, then so is R.*

(2) *If $S = R[X_1, X_2, \ldots, X_n]$ $(n > 0)$ is weakly Arf, then so is R.*

(3) *Let $\varphi : R \to S$ be the faithfully flat homomorphism of rings. If S is a weakly Arf ring, then so is R.*

Proposition 2.5

Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = 1$. Then R is a weakly Arf ring if and only if so is $\widehat{R}$.

Let $A = \mathbb{C}[[t^4, t^5, t^6, s]] \subseteq \mathbb{C}[[t, s]]$. Choose a UFD R s.t. $A \cong \widehat{R}$. Then R is a weakly Arf ring. If $\widehat{R}$ is weakly Arf, then

$$B = \mathbb{C}[[t^4, t^5, t^6]] \to A \cong \widehat{R}$$

ensures that B is weakly Arf, whence B is Arf. This is impossible. Hence $\widehat{R}$ is not weakly Arf.

Theorem 2.6

Suppose that

- *R is an integral domain,*
- *R satisfies (S_2), and*
- *R contains an infinite field.*

Then R is weakly Arf if and only if so is $R[X_1, X_2, \ldots, X_n]$ for $\forall n \geq 1$.

Let $R = k[Y]/(Y^n)$ $(n \geq 1)$ and $S = R[X]$. Then R is weakly Arf and

$$S \text{ is a weakly Arf ring} \iff n \leq 2.$$

Theorem 2.7

Let R be a Noetherian ring, M a finitely generated torsion-free R-module. Then TFAE.

1. *$R \ltimes M$ is a weakly Arf ring.*
2. *R is a weakly Arf ring and M is an $\overline{R}$-module.*

Theorem 2.8

Let $(R, \mathfrak{m}), (S, \mathfrak{n})$ be Noetherian local rings with $k = R/\mathfrak{m} = S/\mathfrak{n}$. Suppose that depth $R > 0$ and depth $S > 0$. Then TFAE.

1. *$R \times_k S$ is a weakly Arf ring.*
2. *R and S are weakly Arf rings.*

3. Strict closures

We define

$$R \subseteq R^* = \{x \in \overline{R} \mid x \otimes 1 = 1 \otimes x \text{ in } \overline{R} \otimes_R \overline{R}\} \subseteq \overline{R}$$

and we say that R is *strictly closed*, if $R = R^*$ holds.

Note that

- R is strictly closed $\Longrightarrow$ R is weakly Arf.

Indeed, let $x, y, z \in R$ with $x \in W(R)$ such that $y/x, z/x \in \overline{R}$. Then

$$\frac{yz}{x} \otimes 1 = \frac{y}{x} \otimes \left(x \cdot \frac{z}{x}\right) = \left(\frac{y}{x} \cdot x\right) \otimes \frac{z}{x} = 1 \otimes \frac{yz}{x}.$$

Example 3.1

- Every Stanley-Reisner algebra is strictly closed.
- Every F-pure Noetherian ring with (S_2) is strictly closed.

Theorem 3.2 (Zariski, Lipman)

Let R be a CM semi-local ring with $\dim R = 1$. Then

(1) *R is strictly closed $\implies$ R is Arf.*

(2) *The converse holds if R contains a field.*

Theorem 3.3

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is strictly closed} \iff R \text{ is Arf.}$$

Theorem 3.4

Let R be a Noetherian ring with (S_2). Then TFAE.

(1) *R is strictly closed.*

(2) *R is weakly Arf, and R_P is Arf for $\forall P \in \operatorname{Spec} R$ with $\operatorname{ht}_R P = 1$.*

Corollary 3.5

Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \geq 2$ and (S_2). Then

$$R \text{ is strictly closed} \iff R \text{ is weakly Arf.}$$

Conjecture 3.6 (Zariski)

Let R be a CM semi-local ring with $\dim R = 1$. Suppose that $\overline{R}$ is a finitely generated R-module. Then the equality

$$\mathrm{Arf}(R) = R^*$$

holds.

- Zariski's conjecture holds if R contains a field (Lipman).

Theorem 3.7

Zariski's conjecture holds.

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Theorem 3.7

Zariski's conjecture holds.

Theorem 3.8

Let R be a CM semi-local ring with $\dim R = 1$. Then

$$R \text{ is Arf} \implies R^G \text{ is Arf}$$

for every finite subgroup G of $\operatorname{Aut} R$ s.t. the order of G is invertible.

Thank you for your attention.